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RADIO NUMBER OF k^{th} POWER OF A PATH

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ABSTRACT. Let G be a connected graph. For any two vertices u and v, let $d(u,v)$ denote the distance between u and v in G. The maximum distance between any pair of vertices is called the diameter of G and is denoted by $diam(G)$. A radio labeling of a connected graph G is an assignment of distinct positive integers to the vertices of G, with $v \in V(G)$ labeled by $f(v)$, such that $|f(u)-f(v)|+d(u,v) \geq 1+diam(G)$ holds for all $u,v \in V$, $u \neq v$. The radio number $rn(f)$ of a radio labeling f of G is the maximum label assigned by f to a vertex of G. The radio number $rn(G)$ of G is the $min\{rn(f)\}$ over all radio labeling f of G. The k^{th} power of a graph G, denoted by G^k, is the graph on the vertices of G with two vertices u and v are adjacent in G^k whenever $d(u,v) \leq k$ in G. In this paper, we completely determine the radio number of the graph P_n^k for positive integers n,k with $k \geq 2$.

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1. INTRODUCTION

All the graphs considered here are undirected, finite, connected and simple. The length of a shortest path between two vertices u and v in a graph G is called the distance between u and v and is denoted by $d_G(u,v)$ or simply $d(u,v)$. We use the standard terminology, the terms not defined here may be found in [1, 10]. The *eccentricity* of a vertex v of a graph G is the distance from the vertex v to a farthest vertex in G. The minimum eccentricity of a vertex in G is the *radius* of G, denoted by $r(G)$, and that of maximum eccentricity of a vertex of G is called the *diameter* of G, denoted by $diam(G)$. A vertex v of G whose eccentricity is equal to the radius of G is a *central vertex*. The k^{th} power of a graph G, denoted by G^k is the graph on the vertices of G with two vertices u and v are adjacent in G^k whenever $d_G(u,v) \leq k$. For any real number x, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Radio labeling of a connected graph G is an injunctive function $f : V(G) \to Z^+$, such that for any two distinct vertices u,v of G, $|f(u) - f(v)| + d(u,v) \geq 1 + diam(G)$ holds. The radio number $rn(f)$ of a radio labeling f of G is the maximum label assigned to a vertex of G. The *radio number* $rn(G)$ of G is the $min\{rn(f)\}$, over all radio labeling f of G. A radio labeling f of G is a *minimal radio labeling* of G if $rn(f) = rn(G)$.

Radio labeling is motivated by the channel assignment problem introduced by Hale et al [5] in 1980. The radio labeling of a graph is most useful in FM radio channel restrictions to overcome the noise effect. The notion of radio labeling was introduced in 2001, by G. Chartrand, David Erwin, F.

Harary and Ping Zhang in [3]. We can find more results on radio labeling of graphs in [4, 6, 8, 9, 11, 12, 14].

D.D.F. Liu and M. Xie obtained radio labeling of square of path in [7]. In 2010, B. Sooryanarayana et al [13] completely determined radio number for the cube of a path. In [2] we obtained radio number for the fourth power of a path. In this paper we generalize the results of [7, 13] and [2].

Throughout this article we denote a path on n vertices by P_n with $V(P_n) = \{v_1, v_2, v_3 \ . \ . \ , v_n\}$ and $E(P_n) = \{v_iv_{i+1} \mid i = 1, 2, ..., n-1\}$. A path on odd number of vertices is called an *odd path* and path on even number of vertices is called an *even path*. An odd path P_{2m+1} has exactly one central vertex namely v_{m+1}, while an even path P_{2m} has two central vertices v_m and v_{m+1}. For an odd path P_{2m+1}, vertices $v_1, v_2, v_3, \dots v_m$ are left vertices and $v_{m+2}, v_{m+3}, v_{m+4}, \dots v_{2m+1}$ are right vertices. Similarly, for an even path P_{2m}, $v_1, v_2, v_3, \dots v_m$ are left vertices and the vertices $v_{m+1}, v_{m+2}, v_{m+3}, \dots v_{2m}$ are right vertices. Two vertices in P_n^k are on same side if both are left vertices or right vertices of P_n, otherwise the vertices are said to be on opposite sides in P_n^k.

For each vertex $u \in V(P_n^k)$, the level of u, denoted by $l(u)$, is the distance in P_n from u to the nearest central vertex of P_n, and the set of levels of the vertices in P_n^k is denoted by $L(V(P_n^k))$. If n is odd, then the set of levels of vertices in P_n^k is $L(V(P_n^k)) = \{\frac{n-1}{2}, \frac{n-1}{2} - 1, \dots, 2, 1, 0, 1, 2, \dots, \frac{n-1}{2} - 1, \frac{n-1}{2}\}$ and if n is even, then $L(V(P_n^k)) = \{\frac{n}{2} - 1, \frac{n}{2} - 2, \dots, 2, 1, 0, 0, 1, 2, \dots, \frac{n}{2} - 2, \frac{n}{2} - 1\}$.

We observe that if v_i and v_j are vertices on opposite sides of the central vertex/vertices, then the distance between v_i and v_j in P_n is given by

$$d_{P_n}(v_i, v_j) = \begin{cases} l(v_i) + l(v_j), & if\ n\ is\ odd \\ l(v_i) + l(v_j) + 1, & if\ n\ is\ even. \end{cases} \tag{1}$$

The main result we prove in this paper is Theorem 1.1. The lower bound is established in Section 2 and a labeling procedure is given in Section 3 to show that the lower bounds achieved in Section 2 are tight.

Theorem 1.1. *For any two positive integers n and k with $2 \leq k \leq n-2$,*

$$rn(P_n^k) = \begin{cases} 2kp^2 + 2, & if\ (m = 0)\ or\ (m = 1\ and\ n < 4k + 1) \\ 2kp^2 + 3, & if\ m = 1\ and\ n \geq 4k + 1 \\ 2kp^2 + 2kp + m + 1, & if\ 2 \leq m \leq k \\ 2kp^2 + 2kp + m, & if\ m = k + 1 \\ 2kp^2 + 4kp + 2k + 2, & if\ k + 2 \leq m \leq 2k - 1, \end{cases}$$

where $p = \lfloor \frac{n}{2k} \rfloor$ and $m = n - 2kp$.

2. Lower Bound

In this section we first discuss some basic properties of the graph P_n^k and then use them to establish the greatest lower bound for radio number of P_n^k.

Observation 1. In the graph P_n^k,

$$d(u, v) = d_{P_n^k}(u, v) = \left\lfloor \frac{d_{P_n}(u, v) + k - 1}{k} \right\rfloor \text{ and } diam(P_n^k) = \left\lfloor \frac{n + k - 2}{k} \right\rfloor. \tag{2}$$

Observation 2. For $n \equiv m\ (mod\ 2k)$, we have $n = 2kp + m$ and hence

$$diam(P_n^k) = \begin{cases} 2p, & if\ m = 0, 1 \\ 2p + 1, & if\ 2 \leq m \leq k + 1 \\ 2p + 2, & if\ k + 2 \leq m \leq 2k - 1. \end{cases} \tag{3}$$

Observation 3. If v_i and v_j are vertices on the same side, then the distance between v_i and v_j in P_n^k is given by

$$d(v_i, v_j) = d_{P_k^n}(v_i, v_j) = \left\lfloor \frac{|l(v_i) - l(v_j) + k - 1|}{k} \right\rfloor.$$

If v_i and v_j are vertices on opposite sides, then by equations (1) and (2), the distance between v_i and v_j in P_n^k is given by

$$d(v_i, v_j) = \begin{cases} \left\lfloor \frac{l(v_i)+l(v_j)+k-1}{k} \right\rfloor, & if\ n\ is\ odd \\ \left\lfloor \frac{l(v_i)+l(v_j)+k}{k} \right\rfloor, & if\ n\ is\ even. \end{cases} \tag{4}$$

Lemma 2.1. *For any two positive integers n and k, with $k \leq n-2$ and $n = 2kp+m$ (where $0 \leq m < 2k$),*

$$rn(P_n^k) \geq \begin{cases} 2kp^2 + 2, & if\ (m = 0)\ or\ (m = 1\ and\ n < 4k + 1) \\ 2kp^2 + 3, & if\ m = 1\ and\ n \geq 4k + 1 \\ 2kp^2 + 2kp + m + 1, & if\ 2 \leq m \leq k \\ 2kp^2 + 2kp + m, & if\ m = k + 1 \\ 2kp^2 + 4kp + 2k + 2, & if\ k + 2 \leq m \leq 2k - 1. \end{cases}$$

Proof. Let f be a radio labeling of the graph P_n^k and $x_1, x_2, \ldots, x_n$ be the sequence of the vertices of P_n^k such that $f(x_{i+1}) > f(x_i)$ for every $i, 1 \leq i \leq n-1$. By the definition of radio labeling, we have

$$f(x_{i+1}) - f(x_i) \geq 1 + diam(P_n^k) - d(x_{i+1}, x_i),\ for\ every\ i,\ 1 \leq i \leq n-1 \tag{5}$$

Summing up all the $n-1$ inequalities in (5), we get

$$\sum_{i=1}^{n-1} [f(x_{i+1}) - f(x_i)] \geq \sum_{i=1}^{n-1} (1 + diam(P_n^k)) - \sum_{i=1}^{n-1} d(x_{i+1}, x_i). \tag{6}$$

The above inequality (6) simplifies to

$$f(x_n) - f(x_1) \geq (n-1)(1 + diam(P_n^k)) - \sum_{i=1}^{n-1} d(x_{i+1}, x_i). \tag{7}$$

If f is a minimal radio labeling of P_n^k, then $f(x_1) = 1$ (Else we can reduce the span of f by $f(x_n) - f(x_1) + 1$ by reducing each label by $f(x_1) - 1$). Therefore, inequality (7) can be written as

$$f(x_n) \geq 1 + (n-1)(1 + diam(P_n^k)) - \sum_{i=1}^{n-1} d(x_{i+1}, x_i). \tag{8}$$

The above inequality (8) implies that $f(x_n)$ is minimum if $\sum_{i=1}^{n-1} d(x_{i+1}, x_i)$ is maximum. Thus we require to determine sequence $\{x_i\}$ of vertices of P_n^k such that $\sum_{i=1}^{n-1} d(x_{i+1}, x_i)$ has maximum value and it can be achieved only if x_i and x_{i+1}, are on opposite sides (follows by equation (4)). Keeping this in mind we now determine a sequence $\{x_i\}$ in different cases as follows.

Case 1: Let n be odd.

Since x_i and x_{i+1} are on opposite sides, by equation (4),

$$d(x_i, x_{i+1}) = \frac{l(x_i) + l(x_{i+1}) + k - 1}{k} - \xi_i, \tag{9}$$

where

$$\xi_i = \frac{l(x_i) + l(x_{i+1}) + k - 1}{k} - \lfloor \frac{l(x_i) + l(x_{i+1}) + k - 1}{k} \rfloor. \tag{10}$$

By denoting $\sum_{i=1}^{n-1} \xi_i = \xi$, from equation (9), we get

$$\begin{aligned}
\sum_{i=1}^{n-1} d(x_{i+1}, x_i) &= \sum_{i=1}^{n-1} \frac{l(x_i) + l(x_{i+1}) + k - 1}{k} - \xi \\
&= \frac{1}{k}\left\{2\sum_{i=1}^{n} l(x_i) - [l(x_1) + l(x_n)]\right\} + \frac{k-1}{k}(n-1) - \xi \\
&= \frac{4}{k}\left\{1 + 2 + \ldots + \frac{n-1}{2}\right\} + \frac{nk - n - k + 1}{k} - \frac{l(x_1) + l(x_n)}{k} - \xi \\
&= \frac{n^2 + 2nk - 2n - 2k + 1}{2k} - (\xi + \eta),
\end{aligned}$$

where $\eta = \frac{l(x_1)+l(x_n)}{k}$. Therefore by equation (8),

$$f(x_n) \geq 1 + (n-1)(1 + diam(P_n^k)) - \frac{n^2 + 2nk - 2n - 2k + 1}{2k} + (\xi + \eta). \tag{11}$$

To minimize $f(x_n)$, we require to obtain sequence $\{x_i\}$, for which $\xi + \eta$ is minimized. This requires obtaining the sequence $\{x_i\}$ with maximum number of pairs $(x_i,\, x_{i+1})$ satisfying $l(x_i) + l(x_{i+1}) \equiv 1(mod\ k)$ so that $\xi_i = 0$ for all such pairs of vertices. For this, we partition the set of vertices $V(P_n^k)$ into subsets $S_{1,0}, S_{0,1}$ and $S_{p,q}$ such that $2 \leq p, q \leq k-1$, $p + q = k + 1$ where $S_{i,j}$ is the subset containing all the left vertices whose level congruent to i modulo k and all the right vertices whose level congruent to j modulo k. S_0 is the subset which contains only the central vertex $v_{\frac{n+1}{2}}$. We observe that for any two vertices v_i and v_j in the same subset $S_{i,j}$ and on opposite sides of the central vertex, the value of ξ_i is 0 and the value of ξ_i is nonzero for vertices v_i and v_j belonging to different subsets. For the sequence $\{x_i\}$ having minimum $\xi + \eta$ (as described in [13]) the following conditions hold;

1. In each $S_{i,j}$, we choose the vertices alternately between the left and right sides of the central vertex.
2. After choosing all the vertices of a subset, the vertices of the next subset is to be chosen.

Thus value of $\xi + \eta$ depends on order in which we choose the subsets $S_{i,j}$. The calculation of minimum value of $\xi + \eta$ for the subcases k odd and k even is discussed below by taking $n \equiv m(mod\ 2k)$, so that $n = 2kp + m$, where $p = \lfloor \frac{n}{2k} \rfloor$.

Subcase 1: k odd .

In this case the partition of $V(P_n^k)$ is $\{S_0, S_{1,0}, S_{0,1}, S_{2,k-1}, S_{k-1,2}, S_{3,k-2}, S_{k-2,3} \cdots S_{\frac{k+1}{2},\frac{k+1}{2}}\}$. Since n is odd, the possible values of m are $1, 3, \ldots, 2k-1$. For $m \leq k$, the subset S_0 and the first $m-1$ subsets $S_{1,0}$, $S_{0,1}$, $S_{2,k-1} \ldots S_{\frac{2k-m+3}{2},\frac{m-1}{2}}$ contain an odd number of vertices and the remaining $k - m + 1$ subsets contain an even number of vertices. Let S_1 be the collection of all subsets containing an odd number of vertices except S_0 and S_2 be the collection of all subsets containing an even number of vertices. The number of subsets in S_1 as well as S_2 depends on m for $m = 1, 3, \ldots 2k-1$, and the number subsets in S_1 (and also in S_2) is the same for the cases $n \equiv m\ (mod\ 2k)$ and $n \equiv 2k - m + 2\ (mod\ 2k)$. Therefore $\xi + \eta$ will be same for the cases $n \equiv m\ (mod\ 2k)$ and $n \equiv 2k - m + 2\ (mod\ 2k)$. Hence it is enough to determine the minimum value of $\xi + \eta$ for $m \leq k$.

Since ξ_i is nonzero for any two consecutive vertices belonging to different subsets, from (10), $\xi_i \geq \frac{1}{k}$. There are $k - m + 1$ subsets in S_2 and hence $\sum \xi_i \geq \frac{k-m}{k}$ for any sequence of all the vertices from subsets in S_2. An arrangement of subsets in S_2, for which $\sum \xi_i = \frac{k-m}{k}$ is shown in Figure 1. Thus to obtain the sequence $\{x_i\}$

with minimum $\xi+\eta$, we fix the arrangement of the subsets of S_2 and then consider the different possibilities of S_0 and S_1.

Let l_s be the level of the starting vertex, and x be the level of the ending vertex in S_1. Then for any sequence $\{x_i\}$ of vertices chosen first all vertices from subsets of S_1, then all the vertices from subsets of S_2, and finally vertex of S_0,

$$\begin{aligned}\xi+\eta &\geq \frac{1}{k}\left[4(1+2+\cdots+\frac{m-1}{2})-(l_s+x)-(m-2)+(x+\frac{m+1}{2}-1)\right]\\ &\quad +\frac{k-m}{k}+\frac{m-1}{2k}+\frac{l_s}{k}\\ &= \frac{m^2-2m+2k+1}{2k}.\end{aligned}$$

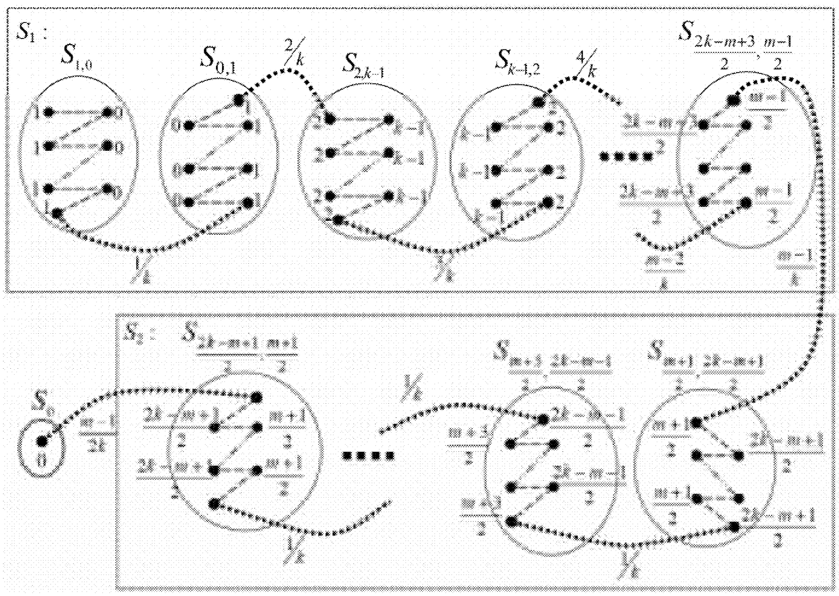

FIGURE 1. Choosing the sequence of vertices $\{x_i\}$ when both n, k are odd.

This shows that the odd sets in S_1 can be chosen in any order. One of such arrangement of subsets (and vertices) of S_1, S_2 and S_0 is shown in Figure 1.

For any sequence $\{x_i\}$ of vertices chosen first all vertices from subsets of S_1, then the vertex of S_0 and all vertices from subsets of S_2,

$$\begin{aligned}\xi+\eta &\geq \frac{1}{k}\left[4\left(1+2+\cdots+\frac{m-1}{2}\right)-(l_s+x)-(m-2)+(x-1)+\frac{m-1}{2}+k-m\right]\\ &\quad +\frac{1}{k}\left[l_s+\frac{m+1}{2}\right]\\ &= \frac{m^2-2m+2k+1}{2k}.\end{aligned}$$

For any sequences $\{x_i\}$ of vertices chosen first all vertices from subsets of S_2, then all vertices from subsets of S_1 and finally the vertex of S_0,

$$\begin{aligned}\xi+\eta &\geq \frac{1}{k}\left[k-m+\frac{m+1}{2}+l_s-1+4(1+2+\cdots+\frac{m-1}{2})-(l_s+x)-(m-2)\right]\\ &\quad+\frac{x-1}{k}+\frac{m+1}{2k}\\ &= \frac{m^2-2m+2k+1}{2k}.\end{aligned}$$

Also it is observed that for sequences $\{x_i\}$ of vertices chosen such that all vertices of S_2 (or vertex of S_0 or all vertices of S_2 and S_0) between any two subsets of S_1, $\xi+\eta \geq \frac{m^2-2m+2k+1}{2k}$. Thus for any sequence $\{x_i\}$, we have,

$$\xi+\eta \geq \tfrac{m^2-2m+2k+1}{2k}.$$

Hence by equation (11), for $m \leq k$ we have,

$$\begin{aligned}f(x_n) &\geq 1+(n-1)(1+diam(P_n^k))-\frac{n^2+2nk-2n-2k+1}{2k}\\ &\quad+\frac{m^2-2m+2k+1}{2k}\\ &\geq 1+(n-1)(1+diam(P_n^k))-\frac{n^2-m^2+2nk-2n+2m-4k}{2k}.\end{aligned}$$

Since the number of subsets $S_{i,j}$ containing an odd number of vertices is the same for the cases $n \equiv m\ (mod\ 2k)$ and $n \equiv 2k-m+2\ (mod\ 2k)$, for the cases $m \geq k+2$ the minimum value of $\xi+\eta$ is same as its value in previous case with m replaced by $2k-m+2$. Hence,

$$\begin{aligned}\xi+\eta &\geq \frac{(2k-m+2)^2-2(2k-m+2)+2k+1}{2k}\\ &= \frac{4k^2+m^2-4km+6k-2m+1}{2k}.\end{aligned}$$

Now by equation (11), for $m \geq k+2$ we have,

$$\begin{aligned}f(x_n) &\geq 1+(n-1)(1+diam(P_n^k))-\frac{n^2+2n(k-1)-2k+1}{2k}\\ &\quad+\frac{4k^2+m^2-4mk+6k-2m+1}{2k}\\ &\geq 1+(n-1)(1+diam(P_n^k))-\frac{n^2-m^2-2(2k^2-nk-2mk+n-m+4k)}{2k}.\end{aligned}$$

Substituting the value for $diam(P_n^k)$ from equation (3) and simplifying the above using $n=2kp+m$, we get

$$f(x_n) \geq \begin{cases} 2kp^2+2, & if\ m=0,1\\ 2kp^2+2kp+m+1, & if\ 2\leq m\leq k\\ 2kp^2+4kp+2k+2, & if\ k+2\leq m\leq 2k-1.\end{cases} \tag{12}$$

Subcase 2: k even.

In this case the partition of $V(P_n^k)$ is $\{S_0,\ S_{1,0},\ S_{0,1},\ S_{2,k-1},\ S_{k-1,2},\ S_{3,k-2},\ S_{k-2,3}\cdots\ S_{\frac{k+2}{2},\frac{k}{2}}\}$. The first m subsets namely $S_0,\ S_{1,0},\ S_{0,1},\ S_{2,k-1},\ S_{k-1,2},\ S_{3,k-2},\ S_{k-2,3}\cdots S_{\frac{m-1}{2},\frac{2k-m+1}{2}},\ S_{\frac{2k-m+1}{2},\frac{m-1}{2}}$ contains an odd number of vertices and remaining subsets contains an even number of vertices. As in the previous case, let S_1 be the collection of all subsets containing an odd number of vertices except S_0 and S_2 be the collection of all subsets containing an even number of vertices.

The number of subsets in S_1 as well as S_2 depends on m for $m = 1, 3, \ldots 2k-1$, but always even. The number of subsets in S_1 (or in S_2) is the same for the cases $n \equiv m \ (mod\ 2k)$ and $n \equiv 2k-m+2 \ (mod\ 2k)$, except for $m = k+1$. When $m = k+1$ all subsets are odd and hence S_2 is an empty set. Hence $\xi + \eta$ will be same for the cases $n \equiv m (mod\ 2k)$ and $n \equiv 2k - m + 2 (mod\ 2k)$. Therefore it is enough to determine minimum value of $\xi + \eta$ for $m \leq k-1$ and $m = k+1$.

For $m \leq k-1$, as similar to the arguments in Subcase 1, for any sequence $\{x_i\}$ of vertices, $\xi + \eta \geq \frac{m^2-2m+2k+1}{2k}$.

Hence by equation (11), for the $m \leq k-1$ we have

$$\begin{aligned} f(x_n) &\geq 1 + (n-1)(1 + diam(P_n^k)) - \frac{n^2 + 2n(k-1) - 2k + 1}{2k} \\ &\quad + \frac{m^2 - 2m + 2k + 1}{2k} \\ &\geq 1 + (n-1)(1 + diam(P_n^k)) - \frac{n^2 - m^2 + 2nk - 2n + 2m - 4k}{2k}. \end{aligned}$$

For $m = k+1$ all the subsets $S_{i,j}$ contains an odd number of vertices. For sequence $\{x_i\}$ which ends with the vertex of S_0 (one of such arrangement is shown in Figure 2), let l_s be the level of starting vertex and x be the level of ending vertex in S_1 then,

$$\begin{aligned} \xi + \eta &\geq \frac{1}{k}\left[4(1 + 2 + \cdots + \frac{m-1}{2} - (l_s + x) - (m-2) + x - 1 + l_s\right] \\ &= \frac{m^2 - 2m + 1}{2k}. \end{aligned}$$

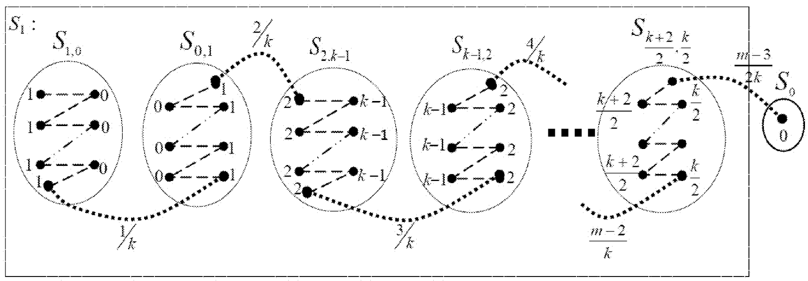

FIGURE 2. Choosing the sequence of vertices $\{x_i\}$ when n is odd, k even and $m = k+1$.

For the sequence $\{x_i\}$ of vertices chosen from the arrangement in which S_0 in between two subsets of $S_{i,j}$, let l_s be the level of starting vertex, x_1 and x_2 be the levels of preceding and succeeding vertices of the vertex of S_0 repetitively and l_e be the level of the last vertex in the sequence. Then,

$$\begin{aligned} \xi + \eta &\geq \frac{1}{k}\left[4(1 + 2 + \cdots + \frac{m-1}{2} - (l_s + x_1 + x_2 + l_e) - (m-1) + (x_1 + x_2)\right] \\ &\quad + \frac{l_s + l_e}{k} \\ &= \frac{m^2 - 2m + 1}{2k}. \end{aligned}$$

Hence in this case for any sequence $\{x_i\}$, $\xi + \eta \geq \frac{m^2-2m+1}{2k}$.
Now by equation (11), for $m = k+1$ we have,

$$\begin{aligned} f(x_n) &\geq 1 + (n-1)(1 + diam(P_n^k)) - \frac{n^2 + 2n(k-1) - 2k + 1}{2k} + \frac{m^2 - 2m + 1}{2k} \\ &\geq 1 + (n-1)(1 + diam(P_n^k)) - \frac{n^2 - m^2 + 2nk - 2n + 2m - 2k}{2k}. \end{aligned}$$

For $m \geq k+3$ the minimum value of $\xi + \eta$ is the same as its value in the case $m \leq k-1$ with m is replaced by $2k - m + 2$. Hence,

$$\begin{aligned} \xi + \eta &\geq \frac{(2k - m + 2)^2 - 2(2k - m + 2) + 2k + 1}{2k} \\ &= \frac{(2k - m)^2 + 6k - 2m + 1}{2k}. \end{aligned}$$

Now by equation (11), for $m \geq k+3$ we have,

$$\begin{aligned} f(x_n) &\geq 1 + (n-1)(1 + diam(P_n^k)) - \frac{n^2 + 2n(k-1) - 2k + 1}{2k} \\ &\quad + \frac{4k^2 + m^2 - 4mk + 6k - 2m + 1}{2k} \\ &\geq 1 + (n-1)(1 + diam(P_n^k)) - \frac{n^2 - m^2 - 4k^2 + 2nk + 4mk - 2n + 2m - 8k}{2k}. \end{aligned}$$

Now substituting $n = 2kp + m$, with $0 \leq m \leq 2k-1$, $p = \lfloor \frac{n}{2k} \rfloor$ and $diam(P_n^k)$ from the equation (3), we get

$$(13) \qquad f(x_n) \geq \begin{cases} 2kp^2 + 2, & if\ m = 0, 1 \\ 2kp^2 + 2kp + m + 1, & if\ 2 \leq m \leq k-1 \\ 2kp^2 + 2kp + m, & if\ m = k+1 \\ 2kp^2 + 4kp + 2k + 2, & if\ k+3 \leq m \leq 2k-1. \end{cases}$$

Case 2: Let n be even.
Since x_i and x_{i+1} are on opposite sides, by equation (4),

$$(14) \qquad d(x_i, x_{i+1}) = \frac{l(x_i) + l(x_{i+1}) + k}{k} - \xi_i,$$

where

$$(15) \qquad \xi_i = \frac{l(x_i) + l(x_{i+1}) + k}{k} - \left\lfloor \frac{l(x_i) + l(x_{i+1}) + k}{k} \right\rfloor.$$

By denoting $\sum_{i=1}^{n-1} \xi_i = \xi$, from equation (14), we get

$$\begin{aligned} \sum_{i=1}^{n-1} d(x_{i+1}, x_i) &= \sum_{i=1}^{n-1} \frac{l(x_i) + l(x_{i+1}) + k}{k} - \xi \\ &= \frac{1}{k}\left\{ 2\sum_{i=1}^{n} l(x_i) - [l(x_1) + l(x_n)] \right\} + (n-1) - \xi \\ &= \frac{4}{k}\left\{ 1 + 2 + \ldots + \frac{n-2}{2} \right\} + (n-1) - \frac{l(x_1) + l(x_n)}{k} - \xi \\ &= \frac{n^2 + 2nk - 2n - 2k}{2k} - (\xi + \eta) \end{aligned}$$

where $\eta = \frac{l(x_1)+l(x_n)}{k}$. Therefore by equation (8),

$$f(x_n) \geq 1 + (n-1)(1 + diam(P_n^k)) - \frac{n^2 + 2nk - 2n - 2k}{2k} + (\xi + \eta) \tag{16}$$

Now $f(x_n)$ is minimized if $\xi+\eta$ is minimized and this requires obtaining sequence $\{x_i\}$ with the maximum number of pairs $(x_i,\ x_{i+1})$ satisfying $l(x_i) + l(x_{i+1}) \equiv 0\ (mod\ k)$ so that $\xi_i = 0$ for such pair of vertices. For this, we partition the set of vertices $V(P_n^k)$ into subsets $S_{0,0}, S_{p,q}$, for all positive integers p and p with $p+q = k$. Where $S_{i,j}$ is the subset containing all left vertices of level congruent to i modulo k and all the right vertices of level congruent to j modulo k. We observe that for any two vertices v_r, v_s in the same subset $S_{i,j}$ on the opposite sides of the central vertex the value of ξ_i is 0 and the value of ξ_i is nonzero for vertices v_r and v_s belonging to different subsets. For the sequence $\{x_i\}$ having minimum $\xi + \eta$ (as described in [13]) the following conditions hold:

1. In each $S_{i,j}$, we choose the vertices alternately between the left and right sides of the central vertex.
2. After choosing all the vertices of a subset, the vertices of the next subset is to be chosen.

Thus the value of $\xi+\eta$ depends on the order in which we choose the subsets $S_{i,j}$. The calculation of the minimum value of $\xi+\eta$ for the subcases k odd and k even is discussed below by taking $n \equiv m(mod\ 2k)$, so that $n = 2kp + m$, where $p = \lfloor \frac{n}{2k} \rfloor$.

Subcase 1: k odd.

In this case partition of set $V(P_n^k)$ is $\{S_{0,0}, S_{k-1,1}, S_{1,k-1}, S_{k-2,2}, \cdots S_{\frac{m-2}{2},\frac{2k-m+2}{2}}, S_{\frac{2k-m}{2},\frac{m}{2}}, S_{\frac{2k-m-2}{2},\frac{m+2}{2}}, \cdots S_{\frac{m}{2},\frac{2k-m}{2}}\}$. Among these subsets, $S_{k-1,1}$, $S_{1,k-1}$, $S_{k-2,2}$, $S_{2,k-2}$, $\cdots$ $S_{\frac{m-2}{2},\frac{2k-m+2}{2}}$ and $S_{\frac{2k-m+2}{2},\frac{m-2}{2}}$ contain an odd number of vertices and the remaining subsets contain an even number of vertices. Let S_1 be the collection of all subsets containing an odd number of vertices and the subset $S_{0,0}$, and S_2 be the collection of all subsets containing an even number of vertices except $S_{0,0}$. The number of subsets in S_1 and S_2 depend on m for $m = 0, 2, \ldots, 2k-2$, but always even. The number of subsets in each of S_1 (and also in S_2) is the same for the cases $n \equiv m(mod\ 2k)$ and $n \equiv 2k - m + 2(mod\ 2k)$. Also $\xi+\eta$ will be same for the cases $n \equiv m(mod\ 2k)$ and $n \equiv 2k - m + 2(mod\ 2k)$. Hence we determine minimum value of $\xi + \eta$ for $m = 0, 2, \ldots, k-1$ and $m = k+1$.

For $m \leq k-1$, we observe S_2 has $k-m+1$ subsets and these have to be arranged in the order as shown in the Figure 3, since for the two connecting vertices in the consecutive subsets (i.e for x_i, x_{i+1}, belong to consecutive subsets of S_2) the value of $\xi_i = \frac{1}{k}$ which is the minimum possible. For the sequence $\{x_i\}$ obtained by the arrangement subsets in S_1 then S_2(one of such arrangement is shown in Figure 3), let l_s be the level of starting vertex and x be the level of ending vertex in S_1 then,

$$\begin{aligned}
\xi + \eta &\geq \frac{1}{k}\left[4(1 + 2 + \cdots + \frac{m-2}{2}) - (l_s + x) + (x + \frac{m+1}{2}) + k - m + l_s + \frac{m}{2}\right] \\
&= \frac{m^2 - 2m + 2k}{2k}.
\end{aligned}$$

The same value is obtained for the choice of S_2 between any two subsets of S_1. Hence by equation (16), for $m \leq k-1$ we have,

$$\begin{aligned}
f(x_n) &\geq 1 + (n-1)(1 + diam(P_n^k)) - \frac{n^2 + 2n(k-1) - 2k}{2k} + \frac{m^2 - 2m + 2k}{2k} \\
&\geq 1 + (n-1)(1 + diam(P_n^k)) - \frac{n^2 - m^2 + 2nk - 2n + 2m - 4k}{2k}.
\end{aligned}$$

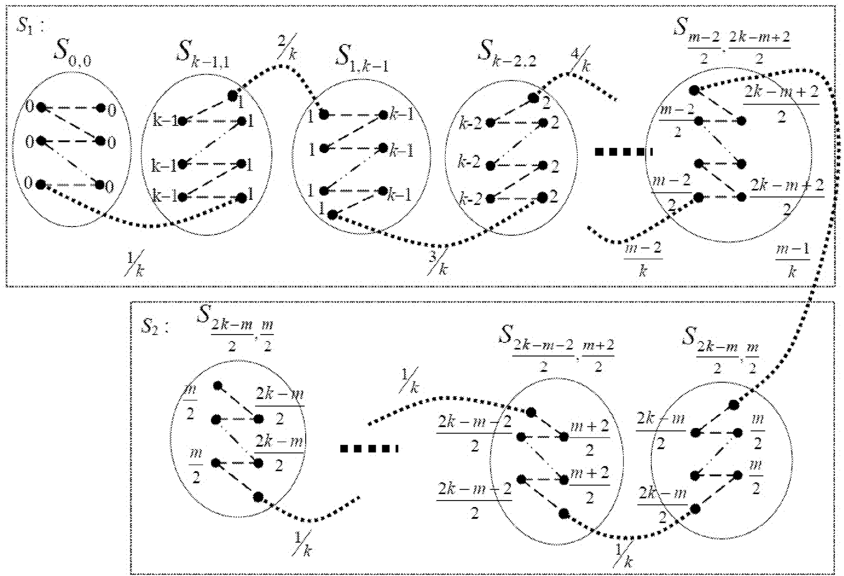

FIGURE 3. Choosing the sequence of vertices $\{x_i\}$ when n is even, k is odd and $m \leq k-1$.

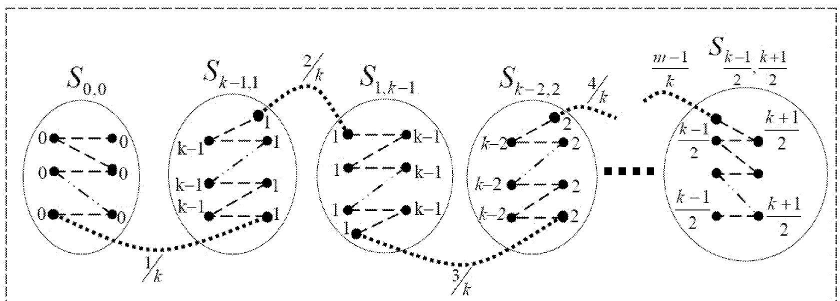

FIGURE 4. Choosing the sequence of vertices $\{x_i\}$ when n is even, k is odd and $m = k+1$.

For $m = k+1$ all the subsets $S_{i,j}$ except $S_{0,0}$ contains an odd number of vertices, hence S_2 is an empty set. Let l_s and x be the levels of starting and ending vertex of sequence $\{x_i\}$. Then,

$$\begin{aligned}\xi + \eta &\geq \frac{1}{k}\left[4(1+2+\cdots+\frac{m-2}{2}) - (l_s + x)\right] + \frac{l_s + x}{k} \\ &= \frac{m^2 - 2m}{2k}.\end{aligned}$$

Now by equation (16), for $m = k+1$ we have,

$$\begin{aligned}f(x_n) &\geq 1 + (n-1)(1 + diam(P_n^k)) - \frac{n^2 + 2n(k-1) - 2k}{2k} + \frac{m^2 - 2m}{2k} \\ &\geq 1 + (n-1)(1 + diam(P_n^k)) - \frac{n^2 - m^2 + 2nk - 2n + 2m - 2k}{2k}.\end{aligned}$$

For m with $k+3 \le m \le 2k-2$ the minimum value of $\xi+\eta$ is the same as its value in the case $m \le k-1$ with m is replaced by $2k-m+2$. Hence,

$$\begin{aligned} \xi+\eta &\ge \frac{(2k-m+2)^2-2(2k-m+2)+2k}{2k} \\ &= \frac{4k^2+m^2-4mk+6k-2m}{2k}. \end{aligned}$$

Now by equation (16), for $m \ge k+3$ we have,

$$\begin{aligned} f(x_n) &\ge 1+(n-1)(1+diam(P_n^k)) - \frac{n^2+2n(k-1)-2k}{2k} \\ &\quad + \frac{4k^2+m^2-4mk+6k-2m}{2k} \\ &\ge 1+(n-1)(1+diam(P_n^k)) - \frac{n^2-m^2-4k^2+2nk+4mk-2n+2m-8k}{2k}. \end{aligned}$$

Substituting for $diam(P_n^k)$ from equation (3) and simplifying the above using $n = 2kp+m$ we get,

$$(17) \qquad f(x_n) \ge \begin{cases} 2kp^2+2, & if\ m = 0,1 \\ 2kp^2+2kp+m+1, & if\ 2 \le m \le k-1 \\ 2kp^2+2kp+m, & if\ m = k+1 \\ 2kp^2+4kp+2k+2, & if\ k+2 \le m \le 2k-2. \end{cases}$$

Subcase 2: k even.

Let $n \equiv m(mod\ 2k)$. In this case partition of $V(P_n^k)$ is $\{S_{0,0},\ S_{k-1,1},\ S_{1,k-1},\ S_{k-2,2},\ \cdots\ S_{\frac{m-2}{2},\frac{2k-m+2}{2}},\ S_{\frac{2k-m}{2},\frac{m}{2}},\ S_{\frac{2k-m-2}{2},\frac{m+2}{2}},\ \cdots\ S_{\frac{m}{2},\frac{2k-m}{2}},\}$. Let S_1 be the collection of all subsets containing an odd number of vertices and the subset $S_{0,0}$. Let S_2 be the collection os all subsets containing an even number of vertices. The number of subsets in S_1 as well as S_2 depends on m for $m = 0,2,\ldots 2k-1$, but always even. The number of subsets in S_1 and also in S_2 is same for the cases $n \equiv m(mod\ 2k)$ and $n \equiv 2k-m+2(mod\ 2k)$. Also $\xi+\eta$ will be same for the cases $n \equiv m(mod\ 2k)$ and $n \equiv 2k-m+2(mod\ 2k)$. Hence it is enough to determine minimum value of $\xi+\eta$ for $m \le k$. As discussed in Subcase 1,

$$\xi+\eta \ge \frac{m^2-2m+2k}{2k}.$$

Hence by equation 16, for $m \le k$ we have,

$$\begin{aligned} f(x_n) &\ge 1+(n-1)(1+diam(P_n^k)) - \frac{n^2+2n(k-1)-2k}{2k} + \frac{m^2-2m+2k}{2k} \\ &\ge 1+(n-1)(1+diam(P_n^k)) - \frac{n^2-m^2+2nk-2n+2m-4k}{2k}. \end{aligned}$$

For $m \ge k+2$ the minimum value of $\xi+\eta$ is obtained by replacing m by $2k-m+2$ in above. Hence,

$$\begin{aligned} \xi+\eta &\ge \frac{(2k-m+2)^2-2(2k-m+2)+2k}{2k} \\ &= \frac{4k^2+m^2-4mk+6k-2m}{2k}. \end{aligned}$$

Now by equation (16), for $m \geq k+2$ we have,

$$\begin{aligned} f(x_n) &\geq 1+(n-1)(1+diam(P_n^k)) - \frac{n^2+2n(k-1)-2k}{2k} \\ &\quad + \frac{4k^2+m^2-4mk+6k-2m}{2k} \\ &\geq 1+(n-1)(1+diam(P_n^k)) - \frac{n^2-m^2-4k^2+2nk+4mk-2n+2m-8k}{2k}. \end{aligned}$$

Substituting for $diam(P_n^k)$ from equation (3) and simplifying using $n = 2kp+m$ we get,

$$(18) \qquad f(x_n) \geq \begin{cases} 2kp^2+2, & if\ m=0,1 \\ 2kp^2+2kp+m+1, & if\ 2 \leq m \leq k \\ 2kp^2+4kp+2k+2, & if\ k+2 \leq m \leq 2k-2. \end{cases}$$

For the case $m = 1$ and $n \geq 4k+1$, we show that the lower bound for $f(x_n)$ is $2kp^2+3$.

When $m = 1$ and $n \geq 4k+1$ we consider the induced subgraph which contains those vertices (of P_n^k) with level congruent to 0 or 1 modulo $2k$ (i.e. subgraph induced by $S_0 \cup S_{0,1} \cup S_{1,0}$). This will be the graph P_N^2 where $N = \frac{2n+k-2}{k}$. Then for $n \geq 4k+1$ we find $N \geq 9$, hence as in [7], the lower bound for $f(x_n)$ increases by 1. Thus, $f(x_n) \geq 2kp^2+3$.

By combining the Inequalities (12),(13), (17)and (18) we get,

$$rn(P_n^k) \geq \begin{cases} 2kp^2+2, & if\ (m=0)\ or\ (m=1\ and\ n<4k+1) \\ 2kp^2+3, & if\ m=1\ and\ n \geq 4k+1 \\ 2kp^2+2kp+m+1, & if\ 2 \leq m \leq k \\ 2kp^2+2kp+m, & if\ m=k+1 \\ 2kp^2+4kp+2k+2, & if\ k+2 \leq m \leq 2k-1. \end{cases}$$

□

3. Upper bound and a Radio labeling

By Lemma 2.1, to establish Theorem 1.1, it suffices to give a radio labeling f of P_n^k with desired span. For this we make use of the following lemma.

Lemma 3.1. *Let P_n^k be the k^{th} power of path on n vertices, where $3 \leq k \leq n-2$, and $\{x_i\}$ for $1 \leq i \leq n$, be a sequence of vertices of P_n^k, such that for $1 \leq i \leq n-2$, the following conditions are satisfied:*

$$(19) \qquad d(x_i, x_{i-1}) + d(x_{i-1}, x_{i-2}) - d(x_{i-2}, x_i) \leq 1 + diam(P_n^k)$$

and

$$(20) \qquad d(x_i, x_{i-1}) + d(x_{i-1}, x_{i-2}) + d(x_{i-2}, x_{i-3}) \leq 2(1 + diam(P_n^k)).$$

Then $f : V(P_n^k) \to \{1, 2, 3, \ldots\}$ defined by

$$(21) \qquad f(x_1) = 1\ and\ \ f(x_{i+1}) = 1 + diam(P_n^k) + f(x_i) - d(x_i, x_{i+1})$$

is a radio labeling.

Proof. It suffices to show that, for $2 \leq i \leq n$, $f(x_i) - f(x_j) \geq 1 + diam(P_n^k) - d(x_i, x_j)$ for any $1 \leq j \leq i-2$.

Case 1 : Let $j = i-2$.

Then by the conditions (19) and (21) we have,

$$\begin{aligned} f(x_i) - f(x_{i-2}) &= [f(x_i) - f(x_{i-1})] + [f(x_{i-1}) - f(x_{i-2})] \\ &= [1 + diam(P_n^k) - d(x_i, x_{i-1})] + [1 + diam(P_n^k) - d(x_{i-1}, x_{i-2})] \\ &= 2[1 + diam(P_n^k)] - [d(x_i, x_{i-1}) + d(x_{i-1}, x_{i-2})] \\ &\geq 2[1 + diam(P_n^k)] - [1 + diam(P_n^k) + d(x_i, x_{i-2})] \\ &\geq 1 + diam(P_n^k) - d(x_i, x_{i-2}). \end{aligned}$$

Case 2 : Let $j = i - 3$.
In this case, by (21) we have,

$$\begin{aligned} f(x_i) - f(x_{i-3}) &= [f(x_i) - f(x_{i-1})] + [f(x_{i-1}) - f(x_{i-2})] + [f(x_{i-2}) - f(x_{i-3})] \\ &= [1 + diam(P_n^k) - d(x_i, x_{i-1})] + [1 + diam(P_n^k) - d(x_{i-1}, x_{i-2})] \\ &\quad + [1 + diam(P_n^k) - d(x_{i-2}, x_{i-3})] \\ &= 3[1 + diam(P_n^k)] - [d(x_i, x_{i-1}) + d(x_{i-1}, x_{i-2}) + d(x_{i-2}, x_{i-3})]. \end{aligned}$$

By the condition (20), we get $f(x_i) - f(x_{i-3}) \geq 1 + diam(P_n^k) \geq 1 + diam(P_n^k) - d(x_i, x_{i-3})$.
Case 3 : Let $j \leq i - 4$.
Then $f(x_j) \leq f(x_{i-3})$, and hence,

$$\begin{aligned} f(x_i) - f(x_j) &\geq f(x_i) - f(x_{i-3}) \\ &\geq 1 + diam(P_n^k) \geq 1 + diam(P_n^k) - d(x_i, x_j). \end{aligned}$$

This completes the proof of Lemma 3.1. □

Now we obtain a sequence $\{x_i\}$ of vertices of P_n^k satisfying the conditions (19) and (20) in Lemma 3.1, so that the function f defined by, $f(x_1) = 1$ and $f(x_{i+1}) = [1 + diam(P_n^k)] + f(x_i) - d(x_i, x_{i+1})$ $\ for\ 1 \leq i \leq n-1$ is a radio labeling function and then,

$$f(x_n) = 1 + (n-1)(1 + diam(P_n^k)) - \sum_{i=1}^{n-1} d(x_{i+1}, x_i). \qquad (22)$$

Now we consider the following cases for various values of m, where $n \equiv m\ (mod\ 2k)$, $0 \leq m \leq 2k - 1$.
Case 1: $m = 0$
Let $n = 2kp$. We consider the sequence of vertices of the graph P_n^k as shown in Table 1. The distance between two consecutive vertices of the sequence $\{x_i\}$ is shown between the vertices.

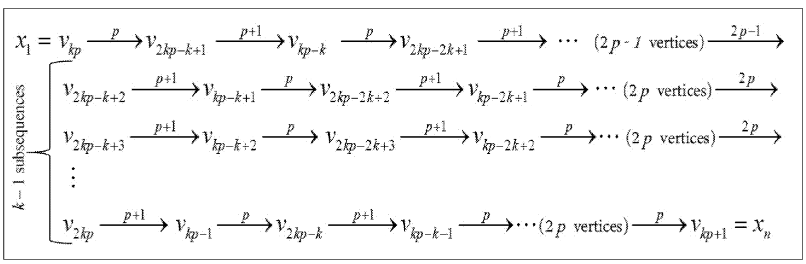

TABLE 1. The sequence $\{x_i\}$ of $V(P_n^k)$ when $n \equiv 0\ (mod\ 2k)$

We observe that the sequence $\{x_i\}$ given in Table 1 satisfies the conditions (19) and (20) in the Lemma 3.1, so that the function $f : V(P_n^k) \to \{1, 2, 3, \ldots\}$ defined by (21) is a radio labeling.For this sequence we observe that,

$$\begin{aligned} \sum_{i=1}^{n-1} d(x_{i+1}, x_i) &= p(p-1) + (p+1)(p-1) + 2p - 1 \\ &+ (k-2)\left[(p+1)p + p(p-1) + 2p\right] + (p+1)p + p(p-1) + p \\ &= 2kp^2 + 2kp - 2p - 2. \end{aligned}$$

Now substituting for $diam(P_n^k)$ and $\sum_{i=1}^{n-1} d(x_{i+1}, x_i)$, in equation (22) we get,

$$f(x_n) = 1 + (2kp-1)(1+2p) - (2kp^2 + 2kp - 2p - 2) = 2kp^2 + 2.$$

Hence in this case we get, $rn(P_n^k) \leq 2kp^2 + 2$.

Case 2: $m = 1$

Let $n = 2kp + 1$. For $n = 2k + 1$ (i.e $p = 1$), we consider the sequence of vertices of the graph P_n^k as shown in Table 2. The distance between two consecutive vertices of the sequence $\{x_i\}$ is shown between the vertices.

$$x_1 = v_k \xrightarrow{2} v_{2k+1} \xrightarrow{2} v_{k-1} \xrightarrow{2} v_{2k} \xrightarrow{2} v_{k-2} \xrightarrow{2}$$
$$v_{2k-1} \xrightarrow{2} v_{k-3} \xrightarrow{2} \cdots\cdots v_1 \xrightarrow{2} v_{k+2} \xrightarrow{1} v_{k+1} = x_n$$

Table 2. The sequence $\{x_i\}$ of $V(P_n^k)$, when $n = 2k + 1$.

We observe that the sequence $\{x_i\}$ given in Table 2 satisfies the conditions (19) and (20) in the Lemma 3.1 so that the function $f : V(P_n^k) \to \{1, 2, 3, \ldots\}$ defined by (21) is a radio labeling. For this sequence we observe that,

$$\sum_{i=1}^{n-1} d(x_{i+1}, x_i) = 2(2k-1) + 1 = 4k - 1.$$

Now substituting for $diam(P_n^k)$ and $\sum_{i=1}^{n-1} d(x_{i+1}, x_i)$, in equation (22) we get,

$$f(x_n) = 1 + (2k)(1+2) - (4k-1) = 2k + 2.$$

Hence in this case we get, $rn(P_n^k) \leq 2k + 2$.

For $n \geq 4k + 1$, we consider the sequence of vertices of the graph P_n^k as shown in Table 3. The distance between two consecutive vertices of the sequence $\{x_i\}$ is shown between the vertices.

We observe that the sequence $\{x_i\}$ given in Table 3, satisfies the conditions (19) and (20) in the Lemma 3.1, so that the function $f : V(P_n^k) \to \{1, 2, 3, \ldots\}$ defined by (21), is a radio labeling. From the Table 3, we observe that,

$$\begin{aligned} \sum_{i=1}^{n-1} d(x_{i+1}, x_i) &= (p-1) + (p+1)(p-1) + p(p-2) + 2p - 1 \\ &+ (k-1)\left[(p+1)p + p(p-1) + 2p\right] + p + 1 \\ &= 2kp^2 + 2kp - 2. \end{aligned}$$

Thus by equation (22), we have

$$\begin{aligned} f(x_n) &= 1 + (n-1)(1 + diam(P_n^k)) - \sum_{i=1}^{n-1} d(x_{i+1}, x_i) \\ &= 1 + (2kp)(1+2p) - (2kp^2 + 2kp - 2) \\ &= 2kp^2 + 3. \end{aligned}$$

$$x_1 = v_{kp+1} \xrightarrow{p-1}$$
$$v_{k+1} \xrightarrow{p+1} v_{kp+k+2} \xrightarrow{p} v_{2k+1} \xrightarrow{p+1} v_{kp+2k+2} \xrightarrow{p} \cdots \text{ (}2p-2\text{ vertices) } \xrightarrow{2p-1}$$

$k-1$ subsequences:

$$v_k \xrightarrow{p+1} v_{kp+k+1} \xrightarrow{p} v_{2k} \xrightarrow{p+1} v_{kp+2k+1} \xrightarrow{p} \cdots \text{ (}2p\text{ vertices) } \xrightarrow{2p}$$
$$v_{k-1} \xrightarrow{p+1} v_{kp+k} \xrightarrow{p} v_{2k-1} \xrightarrow{p+1} v_{kp+2k} \xrightarrow{p} \cdots \text{ (}2p\text{ vertices) } \xrightarrow{2p}$$
$$\vdots$$
$$v_2 \xrightarrow{p+1} v_{kp+3} \xrightarrow{p} v_{k+2} \xrightarrow{p+1} v_{kp+k+3} \xrightarrow{p} \cdots \text{ (}2p\text{ vertices) } \xrightarrow{2p}$$

$$v_1 \xrightarrow{p+1} v_{kp+2} = x_n$$

TABLE 3. The sequence $\{x_i\}$ of $V(P_n^k)$, when $n \equiv 1\ (mod\ 2k)$ and $n \geq 4k+1$.

Hence in this case we get, $rn(P_n^k) \leq 2kp^2 + 3$.

Case 3: $2 \leq m \leq k$.

Let $n = 2kp + m$. If n is odd, then we choose the sequence $\{x_i\}$ of vertices of P_n^k as shown in Table 4, and for n even we choose the sequence $\{x_i\}$ of vertices of P_n^k as shown in Table 5.

$m-1$ subsequences:

$$x_1 = v_{kp+\frac{m-1}{2}} \xrightarrow{p+1} v_{2kp+\frac{m+1}{2}} \xrightarrow{p+2} v_{kp+\frac{m-1}{2}-k} \xrightarrow{p+1} v_{2kp+\frac{m+1}{2}-k} \xrightarrow{p+2} \cdots \text{ (}2p+1\text{ vertices) } \xrightarrow{p+1}$$
$$v_{kp+\frac{m+3}{2}} \xrightarrow{p+1} v_{\frac{m+1}{2}} \xrightarrow{p+2} v_{kp+\frac{m+3}{2}+k} \xrightarrow{p+1} v_{\frac{m+1}{2}+k} \xrightarrow{p+2} \cdots \text{ (}2p+1\text{ vertices) } \xrightarrow{p+1}$$
$$v_{kp+\frac{m-3}{2}} \xrightarrow{p+1} v_{2kp+\frac{m-1}{2}} \xrightarrow{p+2} v_{kp+\frac{m-3}{2}-k} \xrightarrow{p+1} v_{2kp+\frac{m-1}{2}-k} \xrightarrow{p+2} \cdots \text{ (}2p+1\text{ vertices) } \xrightarrow{p+1}$$
$$v_{kp+\frac{m+5}{2}} \xrightarrow{p+1} v_{\frac{m+3}{2}} \xrightarrow{p+2} v_{kp+\frac{m+5}{2}+k} \xrightarrow{p+1} v_{\frac{m+3}{2}+k} \xrightarrow{p+2} \cdots \text{ (}2p+1\text{ vertices) } \xrightarrow{p+1}$$
$$\vdots$$

$k-m+1$ subsequences:

$$v_{kp} \xrightarrow{p+1} v_{2kp+1} \xrightarrow{p+2} v_{kp-k} \xrightarrow{p+1} v_{2kp-k+1} \xrightarrow{p+1} \cdots \text{(}2p\text{ vertices) } \xrightarrow{2}$$
$$v_{kp-1} \xrightarrow{p+1} v_{2kp} \xrightarrow{p+2} v_{kp-k-1} \xrightarrow{p+1} v_{2kp-k} \xrightarrow{p+1} \cdots \text{(}2p\text{ vertices) } \xrightarrow{2}$$
$$\vdots$$
$$v_{kp-k+m} \xrightarrow{p+1} v_{2kp-k+m+1} \xrightarrow{p+2} v_{kp-2k+m} \xrightarrow{p+1} v_{2kp-2k+m+1} \cdots \text{(}2p\text{ vertices) } \xrightarrow{1}$$

$$v_{kp+\frac{m+1}{2}} = x_n$$

TABLE 4. The sequence $\{x_i\}$ of $V(P_n^k)$, when $n \equiv m\ (mod\ 2k)$, $m \leq k$ and n odd.

We observe that the sequences $\{x_i\}$ given in Table 4 and Table 5 satisfy the conditions (19) and (20) in the Lemma 3.1, so that the function $f : V(P_n^k) \to \{1, 2, 3, \dots\}$ defined by (21), is a radio labeling function and for the sequences in

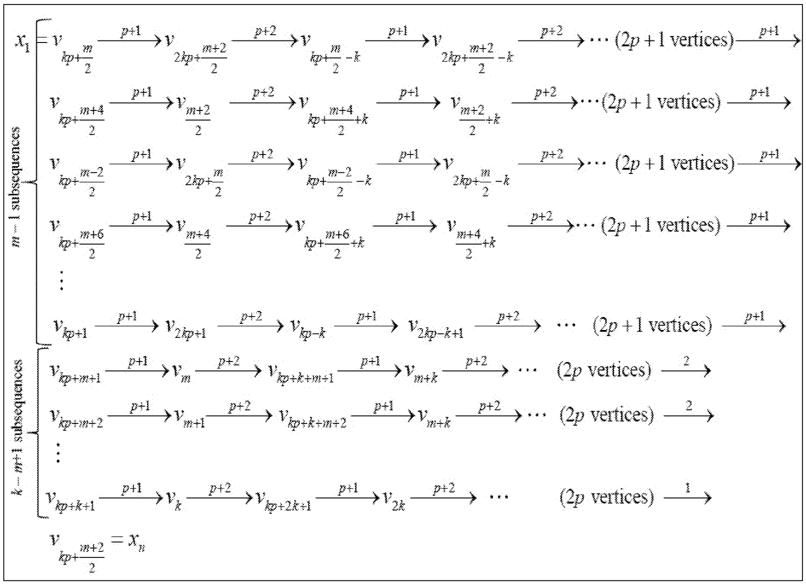

TABLE 5. The sequence $\{x_i\}$ of $V(P_n^k)$, when $n \equiv m \ (mod\ 2k)$, $m \leq k$ and n even.

both the tables we observe that,

$$\begin{aligned}\sum_{i=1}^{n-1} d(x_{i+1}, x_i) &= (m-1)[(p+1)p + (p+2)p + (p+1)] \\ &+ (k-m)[(p+1)p + (p+2)(p-1) + 2] \\ &+ (p+1)p + (p+2)(p-1) + 1 \\ &= 2kp^2 + 2kp + 2pm + m - 2p - 2.\end{aligned}$$

Now substituting for $diam(P_n^k)$ and $\sum_{i=1}^{n-1} d(x_{i+1}, x_i)$, in equation (22) we get,

$$\begin{aligned}f(x_n) &= 1 + (2kp + m - 1)(2p + 2) - (2kp^2 + 2kp + 2pm + m - 2p - 2) \\ &= 2kp^2 + 2kp + m + 1.\end{aligned}$$

Hence in this case we get, $rn(P_n^k) \leq 2kp^2 + 2kp + m + 1$.
Case 4: $m = k + 1$.
Let $n = 2kp + k + 1$. If n is odd, then we choose the sequence $\{x_i\}$ of vertices of P_n^k as shown in Table 6, and for n even we choose the sequence $\{x_i\}$ of vertices of P_n^k as shown in Table 7.

We observe that the sequences $\{x_i\}$ given in Table 6 and Table 7 satisfy the conditions (19) and (20) in the Lemma 3.1, so that the function $f : V(P_n^k) \to \{1, 2, 3, \ldots\}$ defined by (21), is a radio labeling function and for the sequences in both the tables we have,

$$\begin{aligned}\sum_{i=1}^{n-1} d(x_{i+1}, x_i) &= k[(p+1)(p+1) + (p+2)p] \\ &= 2kp^2 + 4kp + k.\end{aligned}$$

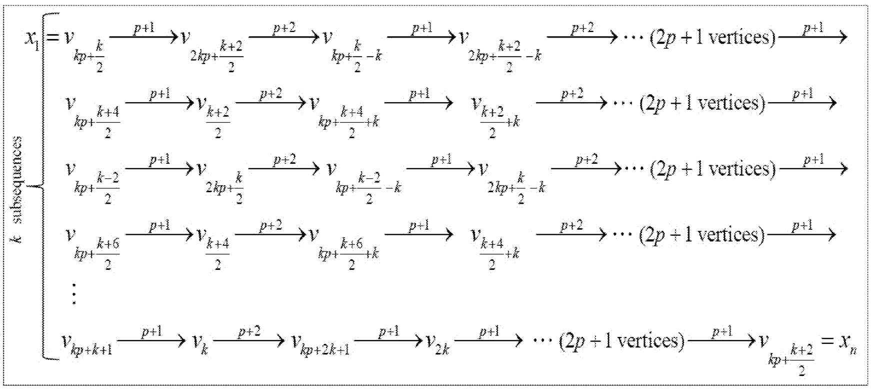

TABLE 6. The sequence $\{x_i\}$ of $V(P_n^k)$ when $n \equiv k + 1\ (mod\ 2k)$ and n odd.

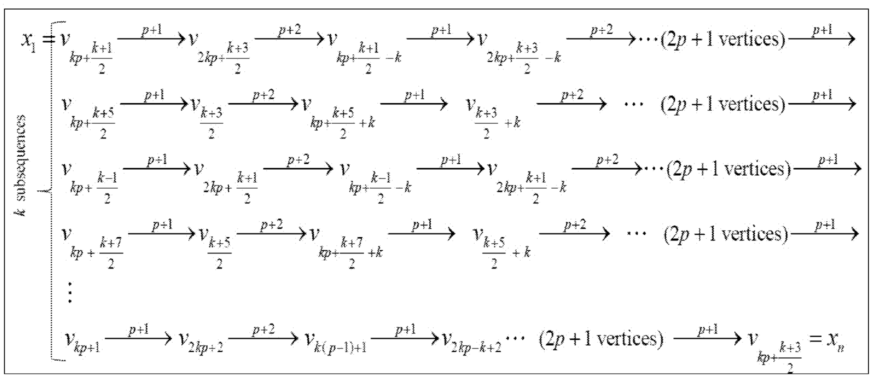

TABLE 7. The sequence $\{x_i\}$ of $V(P_n^k)$ when $n \equiv k + 1\ (mod\ 2k)$ and n even.

Now substituting for $diam(P_n^k)$ and $\sum_{i=1}^{n-1} d(x_{i+1}, x_i)$, in equation (22) we get,

$$f(x_n) = 1 + (2kp + k)(2p + 2) - (2kp^2 + 4kp + k) = 2kp^2 + 2kp + k + 1.$$

Hence in this case we get, $rn(P_n^k) \leq 2kp^2 + 2kp + k + 1$.
Case 5: $k + 2 \leq m \leq 2k - 1$.
Let $n = 2kp + m$. If n is odd, then we choose the sequence $\{x_i\}$ of vertices of P_n^k, as shown in Table 8 and for n even we choose the sequence $\{x_i\}$ of vertices of P_n^k, as shown in Table 9.

We observe that the sequences $\{x_i\}$ given in Table 8 and Table 9 satisfy the conditions (19) and (20) in the Lemma 3.1, so that the function $f : V(P_n^k) \to \{1, 2, 3, \ldots\}$ defined by (21) is a radio labeling function and for the sequences in both the tables we have,

$$\begin{aligned}\sum_{i=1}^{n-1} d(x_{i+1}, x_i) &= (2k - m + 1)[(p + 1)(p + 1) + (p + 2)p] + p \\ &\quad +(m - k - 1)[(p + 2)(p + 1) + (p + 1)p + 2p + 2] - (p + 1) \\ &= 2kp^2 + 2kp + 2pm + 3m - 2p - 2k - 4.\end{aligned}$$

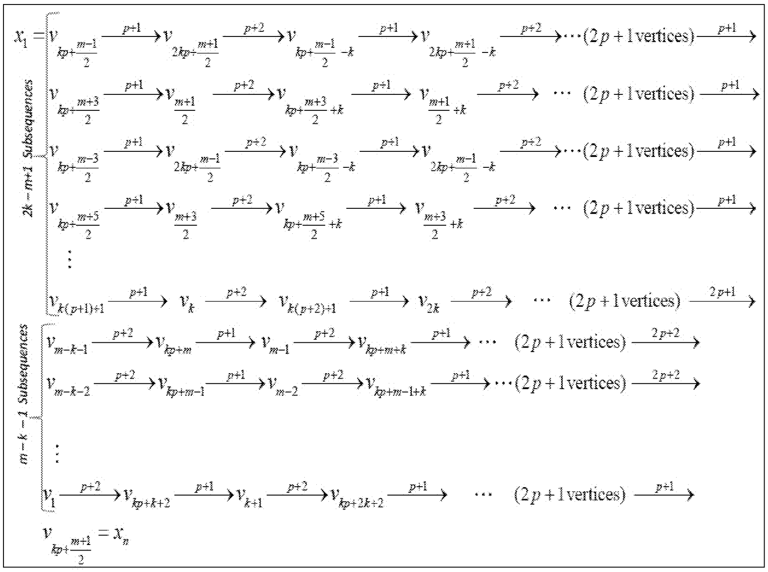

TABLE 8. The sequence $\{x_i\}$ of $V(P_n^k)$ when $n \equiv m(mod\ 2k)$, $m \geq k+2$ and n odd.

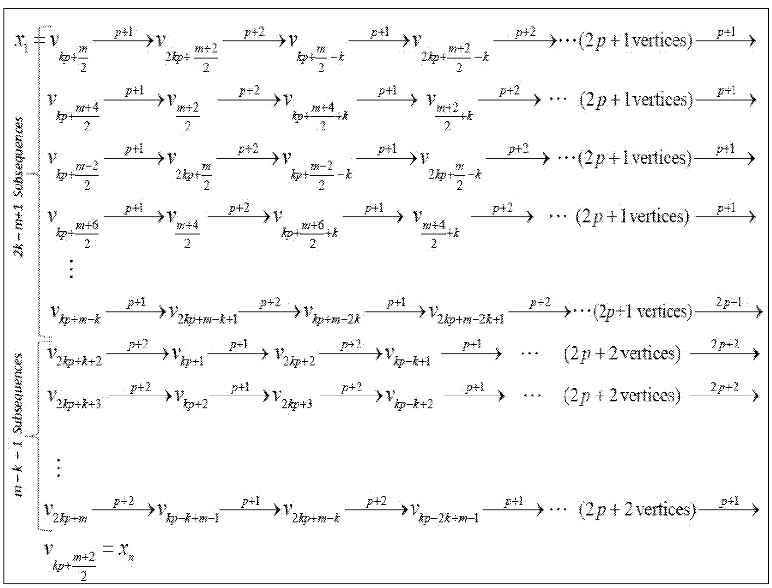

TABLE 9. The sequence $\{x_i\}$ of $V(P_n^k)$ when $n \equiv m\ (mod\ 2k)$, $m \geq k+2$ and n even.

Now substituting for $diam(P_n^k)$ and $\sum_{i=1}^{n-1} d(x_{i+1}, x_i)$, in equation (22) we get,

$$\begin{aligned} f(x_n) &= 1 + (2kp + m - 1)(2p + 3) - (2kp^2 + 2kp + 2pm + 3m - 2p - 2k - 4) \\ &= 2kp^2 + 4kp + 2k + 2. \end{aligned}$$

Hence in this case we get,$rn(P_n^k) \leq 2kp^2 + 4kp + 2k + 2$.

Combining all above cases we conclude that; For any two positive integers n and k, with $k \leq n-2$ and $n = 2kp + m$ (where $0 \leq m < 2k$),

$$rn(P_n^k) \leq \begin{cases} 2kp^2 + 2, & if\ (m=0)\ or\ (m=1\ and\ n < 4k+1) \\ 2kp^2 + 3, & if\ m = 1\ and\ n \geq 4k+1 \\ 2kp^2 + 2kp + m + 1, & if\ 2 \leq m \leq k \\ 2kp^2 + 2kp + k + 1, & if\ m = k+1 \\ 2kp^2 + 4kp + 2k + 2, & if\ k+2 \leq m \leq 2k-1. \end{cases}$$

This completes the proof of the Theorem 1.1. □

Remark 3.2. *The statement of the Theorem 1.1 also holds good for $k = n-1$. In this case $p = 0$, so $m = n$ and $rn(P_n^k) = 2kp^2 + 2kp + m = n$. This is valid by the fact that $rn(K_n) = n$ and $P_n^{n-1} \equiv K_n$.*

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References

[1] Buckley F and Harary F, *Distance in Graphs*, Addison-Wesley,(1990).

[2] P. Devadas Rao, B. Sooryanarayana and Chandru Hegde, *Radio labeling of fourth power of a path*, International Journal of Combinatorial Graph Theory and Applications Vol 3, No. 2.(July-December 2010), 81-103.

[3] G. Chartrand, D. Erwin, F. Harary and P. Zhang, *Radio labelings of graphs*, Bull. Inst. Combin. Appl., 33 (2001), 77-85.

[4] Gary Chartrand, David Erwin, Ping Zhang, *A graph labeling problem Suggested by FM Channel Restrictions*, Bulletin of the Inst. Combin. Appl. 43 (2005), 43-57.

[5] W. K. Hale *Frequency assignment : Theory and applications*, Proc. IEEE 68 (1980) 14971514.

[6] D.D.F. Liu and M. Xie, *Radio number for square of cycles*, congr, Numer., 169 (2004) 105-125.

[7] D.D.F. Liu and M. Xie, *Radio number of square paths*, Ars Comb., 90(2009), 307-319

[8] D.D. F. Liu and Xuding Zhu, *Multilevel Distance Labelings for paths and cycles*, SIAM J. Discrete Math., 19 (2005) 610-621.

[9] J. A. Gallian, *A dynamic survey of graph labeling*, The Electronic Journal of Combinatorics, # DS6,(2009),1-219.

[10] Hartsfield Gerhard and Ringel, *Pearls in Graph Theory*, Academic Press, USA, 1994.

[11] B. Sooryanarayana and Raghunath.P, *Radio labeling of cube of a cycle*, Far East J. Appl. Math 29(1)(2007), 113-147.

[12] B. Sooryanarayana and Raghunath.P, *Radio labeling of C_n^4*, Journal of Applied Mathematical Analysis and Applications 3 (2007), no. 2, 195-228.

[13] B. Sooryanarayana, Vishukumar M and Majula K, *Radio labeling of cube of a Path*, International J. Math. Comb. 1 (2010), 5-29.

[14] J.Van den Heuvel, R.Leese and M.Shepherd, *Graph labeling and radio Channel Assignment*, J.Graph Theory, 29(1998), 263-283.

[15] P.Zhang, *Radio labellings of Cycles*, Ars Combin., 65 (2002), 21-32.

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